

APPENDIX

MATHEMATICS OF DISTORTION PRODUCT OTOACOUSTIC EMISSION GENERATION: A TUTORIAL

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This appendix is designed to be a tutorial dealing with the mathematics of distortion products. Distortion products are typically encountered in audiology through the study of otoacoustic emissions. However, recently it has been found that auditory evoked steady-state responses can also produce distortion products (Lins & Picton 1995). In general, distortion products can be observed in any nonlinear system that is forced to oscillate.

In this tutorial I address the following questions:

- How are distortion products produced?
- How are distortion products related to nonlinearities?
- How can distortion products be generated by transducers and microphones when they do not have hair cells?
- What do "Quadratic" and "Cubic" mean and how are these terms related to $f_2 - f_1$ and $2f_1 - f_2$ distortion products?
- Are there any other kinds of distortion products?

The mathematical description of distortion products will only require some basic algebra and trigonometry. The best way to approach this tutorial is to set aside a half hour and follow along using a pencil and paper. My intention is to aid you in making connections between the mathematical and physical concepts.

LINEAR RESPONSES

Many systems oscillate: masses on springs (or rubber bands), branches in the wind, your vocal cords, guitar strings, waves on the surface of a lake, and the basilar membrane in the inner ear. Each of these systems has frequencies at which they prefer to oscillate. These frequencies are called the natural frequencies or resonance frequencies of the system. If we take any one of these systems and try to force it to oscillate at a given frequency, perhaps by shaking it, it may oscillate at a large amplitude or a small amplitude. This response amplitude depends on the relationship between the driving frequency (the frequency of our shaking) and the resonance or natural frequency of the system.

If the system is linear and we drive it with a sine wave of a given frequency, we find that it will oscillate at that frequency. Mathematically we can write the response of the system as:

$$R(t) = A(f) B \sin(2\pi ft + \phi(f)), \quad (1)$$

where $R(t)$ is the response or position of the system as a function of time, t is the elapsed time, f is the frequency of the stimulation or the driving frequency, $A(f)$ is a function describing how well the system responds to the driving frequency, $\phi(f)$ is a function describing how the phase of the response depends on the driving frequency, and B is the amplitude of the stimulation. In the equation above, we have ignored the terms describing how the response grows as it is stimulated. Instead, we will focus on the steady-state response or the long-term behavior of the system.

The idea is simple. The response of the system is essentially a sine wave at the same frequency as the driving frequency, but it is phase-shifted by an amount described by $\phi(f)$. Its amplitude depends on the amplitude of the stimulation, B , but it also depends on how well the system responds to that frequency, $A(f)$. If the system does not respond well at that frequency then $A(f)$ may be quite small and the response amplitude will be quite a bit smaller than the amplitude of stimulation, B . But if we drive it close to the resonance frequency, then $A(f)$ may be large and the response may become larger than the stimulation amplitude. This is essentially what happens when you push someone on a swing, as long as you push at the right frequency. The function $A(f)$ can also be visualized in terms of the basilar membrane in the cochlea. Different parts of the basilar membrane oscillate best at different frequencies and we can describe the response characteristics of each point on the basilar membrane with a function $A(f)$.

To improve the readability of the equations we will change the notation a bit. Instead of repeatedly writing $2\pi f$, we will introduce what is called the angular frequency, ω , where $\omega = 2\pi f$. This way we can write the sine function, $\sin(2\pi ft)$, above as $\sin(\omega t)$. Also we will write the frequency response as $A(\omega)$ instead of $A(f)$. Note that the functions, $A(\omega)$ and $A(f)$, are not quite the same function (actually $A(\omega) = A(2\pi f)$), but we will call both functions A just to keep the notation simple. From now on we will ignore the phase by assuming that $\phi(f)$ is zero. With these changes, equation (1) simplifies to

$$R(t) = A(\omega) B \sin(\omega t). \quad (2)$$

The point that I want to make in this section is that the system above is linear. By linear I mean that if one doubles the amplitude of the stimulation, the magnitude of the response amplitude doubles, if one triples the stimulation amplitude, the magnitude of the response amplitude triples, and so on. In addition, if one stimulates the system with two frequencies, ω_1 and ω_2 , simultaneously, say with

$$B \sin(\omega_1 t) + C \sin(\omega_2 t), \quad (3)$$

then the response will look like

$$R(t) = A(\omega_1) B \sin(\omega_1 t) + A(\omega_2) C \sin(\omega_2 t). \quad (4)$$

You can see a trend here. This is just the sum of the responses of the system when it is being driven at each frequency separately. This is also what is meant by linear. The equation shows that the response has two frequencies present and these are precisely the frequencies at which the system is being driven. The only difference is that their amplitudes are affected differently depending on the frequency response of the system. Doubling the amplitude of one of the stimulation frequencies will result in a doubling of the response amplitude at that frequency only.

At this point you should take a moment and make sure it is clear to you that the response in equation (4) consists of two frequencies, ω_1 and ω_2 . Just read the equation and look at the sine functions that are being summed. There is one sine wave of frequency ω_1 plus one sine wave of frequency ω_2 . Note that the sine waves could be replaced with cosines and one would still obtain the same frequencies. This is because a sine wave is a phase-shifted version of a cosine wave.

QUADRATIC NONLINEARITIES

Not all systems have to work like this. What happens if the presence of two frequencies in the driving stimulus affect the response differently than described above? Consider the following response function:

$$R(t) = A(\omega_1) B \sin(\omega_1 t) + A(\omega_2) C \sin(\omega_2 t) + D(\omega_1, \omega_2) B \sin(\omega_1 t) C \sin(\omega_2 t). \quad (5)$$

It is the same as before, but now there is a third term that depends on the product of the two driving sine waves and a new frequency response function, $D(\omega_1, \omega_2)$, that depends on both driving frequencies.

Things are getting a little messy so let's simplify a bit. Say that the frequency response of the system in all cases is $A(\omega) = 1$ and that $D(\omega_1, \omega_2) = 1$. In addition, let's say that the driving stimuli have unit amplitude so that $B = 1$ and $C = 1$. This strips away the unnecessary detail so we can better visualize what is happening. The simplified response becomes

$$R(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + \sin(\omega_1 t) \sin(\omega_2 t). \quad (6)$$

This response depends on the sum of the sine functions at the two driving frequencies and the product of those sine functions. There is no reason that this cannot happen physically, and in fact, complications like this are quite common in approximations of real systems. The last term is called a nonlinear term.

There are several similar nonlinear terms that we could have added (note that 7b is the one above):

$$\sin(\omega_1 t) \sin(\omega_1 t) = \sin^2(\omega_1 t) \quad (7a)$$

$$\sin(\omega_1 t) \sin(\omega_2 t) \quad (7b)$$

$$\sin(\omega_2 t) \sin(\omega_2 t) = \sin^2(\omega_2 t) \quad (7c)$$

These terms have the effect of destroying the linearity of the response. If the amplitude of one of the driving frequencies is doubled, then the response at that frequency is not necessarily doubled. In addition, the response at one frequency may now depend on the response at another frequency. This is called nonlinearity and the system is said to be nonlinear. More specifically, as the nonlinear term consists of the product of two sine functions, the nonlinearity is said to be quadratic (as computing the area of a square or quadrilateral requires the product of two terms, the height and the width). Another example is the quadratic equation, $a x^2 + b x + c = 0$, where the x squared term is a nonlinear term similar to (7a) above. Nonlinearities often imply interactions among the components of the system. In this case, the responses to the two driving oscillations are interacting with one another.

In this simple nonlinear system, what frequencies are found in the response? Well, the response is

$$R(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + \sin(\omega_1 t) \sin(\omega_2 t). \quad (6)$$

It looks like we have an ω_1 from the first sine function and an ω_2 from the second sine function, but the third quadratic term is not a simple sine wave so we cannot just read off the frequencies. We need to do some trigonometry. One can look up the following trigonometric identities (one can also derive these writing the sine waves in terms of exponentials):

$$\sin(A) \sin(B) = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \quad (I1)$$

$$\cos(A) \cos(B) = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \quad (I2)$$

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \quad (I3)$$

$$\cos(A - B) = \cos(B - A) \quad (I4)$$

$$\sin(A - B) = -\sin(B - A) \quad (I5)$$

The first three identities describe how products of sines and cosines are related to sums of sines and cosines. This will be useful in evaluating our product of sine functions. The last two identities describe the symmetry of the sine and cosine functions. You can probably see where this is going. We can use identity (I1) above to deal with our product of sine functions and we will get a sum of cosines. Let's try it. First just look at the third term in the response (6) and use identity (I1) above:

$$\sin(\omega_1 t) \sin(\omega_2 t) = \frac{1}{2} [\cos(\omega_1 t - \omega_2 t) - \cos(\omega_1 t + \omega_2 t)] \quad (8)$$

Since, usually $\omega_2 > \omega_1$, we can rewrite the first cosine, $\cos(\omega_1 t - \omega_2 t)$, using identity (I4) and get $\cos(\omega_2 t - \omega_1 t)$. We can also factor out the t's writing $(\omega_2 t - \omega_1 t)$ as $((\omega_2 - \omega_1) t)$, and do the same for $(\omega_1 t + \omega_2 t)$. The result is

$$\sin(\omega_1 t) \sin(\omega_2 t) = \frac{1}{2} [\cos((\omega_2 - \omega_1)t) - \cos((\omega_1 + \omega_2)t)]. \quad (9)$$

Finally, the response function in equation (6) can be rewritten as

$$R(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + \frac{1}{2} \cos((\omega_2 - \omega_1)t) - \frac{1}{2} \cos((\omega_1 + \omega_2)t). \quad (10)$$

You should follow along with pencil and paper to make sure you get the idea. Now we have a sum of sines and cosines and we can read off the angular frequencies present in the response. These frequencies are ω_1 , ω_2 , $(\omega_2 - \omega_1)$, and $(\omega_1 + \omega_2)$. Replacing ω with $2\pi f$ we find that the frequencies present in the response are f_1 , f_2 , $(f_2 - f_1)$, $(f_1 + f_2)$. We have derived the quadratic distortion products $(f_2 - f_1)$ and $(f_1 + f_2)$. Remember that they come from the quadratic nonlinearity in the response function and are due to an interaction between the responses to both frequencies. For this reason, they are called quadratic distortion products. Note also that their contribution to the response is only one half as much as the linear terms. They are generally not as strong as the responses to the two original frequencies. If you are wondering why sines and cosines are both present in the response, remember that a cosine is a phase-shifted sine wave and vice versa. In addition, the negative amplitude of a sine or cosine can also be interpreted as a phase-shift, as in the fourth term in equation (10) above. I have done my best to keep phase effects out of this tutorial, but in some cases, it is unavoidable.

CUBIC NONLINEARITIES

As you may have already guessed, the response of the system can be even more complicated. Let's look at a response like this:

$$R(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + \sin^2(\omega_1 t) \sin(\omega_2 t) \quad (11)$$

where the third term can be written as $\sin(\omega_1 t) \sin(\omega_1 t) \sin(\omega_2 t)$. It is a product of three sine functions and is called a cubic nonlinearity. The terminology comes from the fact that the volume of a cube is the product of three quantities, its length, its width, and its height.

What is the frequency content of this response? Well, again we can easily read off the frequencies from the first two terms, ω_1 and ω_2 , but the third term again isn't a pure sine or cosine function. We have to expand the cubic term into a sum of sines and cosines again. To expand this let's look at the $\sin^2(\omega_1 t)$ part first. Using identity (I1) again (this is the same as before, but now A and B are both equal to ω_1):

$$\begin{aligned} \sin(\omega_1 t) \sin(\omega_1 t) &= \frac{1}{2} [\cos((\omega_1 - \omega_1)t) - \cos((\omega_1 + \omega_1)t)] \\ &= \frac{1}{2} [\cos(0) - \cos(2\omega_1 t)] \\ &= \frac{1}{2} [1 - \cos(2\omega_1 t)] \\ &= \left[\frac{1}{2} - \frac{1}{2} \cos(2\omega_1 t) \right]. \end{aligned} \quad (15)$$

Substituting equation (15) into the cubic term in equation (11) we find that

$$\sin^2(\omega_1 t) \sin(\omega_2 t) = [\frac{1}{2} - \frac{1}{2} \cos(2\omega_1 t)] \sin(\omega_2 t). \quad (16)$$

We continue using the same procedure as before. Any time we see a product of sines and cosines we use the identities above to turn them into sums of sines or cosines.

$$\begin{aligned} \sin^2(\omega_1 t) \sin(\omega_2 t) &= [\frac{1}{2} - \frac{1}{2} \cos(2\omega_1 t)] \sin(\omega_2 t) \\ &= \frac{1}{2} \sin(\omega_2 t) - \frac{1}{2} \cos(2\omega_1 t) \sin(\omega_2 t) \\ &= \frac{1}{2} \sin(\omega_2 t) - \frac{1}{2} \sin(\omega_2 t) \cos(2\omega_1 t) \end{aligned} \quad (17)$$

Now that we have multiplied it out, there is now a product of a sine and a cosine so we can use identity (I3) to expand it into a sum of sines:

$$\begin{aligned} &= \frac{1}{2} \sin(\omega_2 t) - \frac{1}{2} \{ \frac{1}{2} [\sin(\omega_2 t - 2\omega_1 t) + \sin(\omega_2 t + 2\omega_1 t)] \} \\ &= \frac{1}{2} \sin(\omega_2 t) - \frac{1}{4} \sin(\omega_2 t - 2\omega_1 t) - \frac{1}{4} \sin(\omega_2 t + 2\omega_1 t). \end{aligned} \quad (18)$$

Recall that we can use identity (I5) to set $\sin(\omega_2 t - 2\omega_1 t) = -\sin(2\omega_1 t - \omega_2 t)$, and we know that $\sin(\omega_2 t + 2\omega_1 t) = \sin(2\omega_1 t + \omega_2 t)$ so we get

$$\begin{aligned} &= \frac{1}{2} \sin(\omega_2 t) + \frac{1}{4} \sin(2\omega_1 t - \omega_2 t) - \frac{1}{4} \sin(2\omega_1 t + \omega_2 t) \\ &= \frac{1}{2} \sin(\omega_2 t) + \frac{1}{4} \sin((2\omega_1 - \omega_2)t) - \frac{1}{4} \sin((2\omega_1 + \omega_2)t). \end{aligned} \quad (19)$$

Now we can put the expanded cubic term above (19) back into the response equation (11),

$$R(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + \frac{1}{2} \sin(\omega_2 t) + \frac{1}{4} \sin((2\omega_1 - \omega_2)t) - \frac{1}{4} \sin((2\omega_1 + \omega_2)t), \quad (20)$$

and simplify it by combining the second and third terms to obtain our result:

$$R(t) = \sin(\omega_1 t) + \frac{3}{2} \sin(\omega_2 t) + \frac{1}{4} \sin((2\omega_1 - \omega_2)t) - \frac{1}{4} \sin((2\omega_1 + \omega_2)t). \quad (21)$$

Now we can read off the angular frequencies found in the response. We find ω_1 , ω_2 , $(2\omega_1 - \omega_2)$, and $(2\omega_1 + \omega_2)$, which correspond to the frequencies f_1 , f_2 , $(2f_1 - f_2)$ and $(2f_1 + f_2)$. We have found the cubic distortion products $(2f_1 - f_2)$ and $(2f_1 + f_2)$. Notice that these frequencies are only one fourth of the amplitude of the responses of the original driving frequencies. This is an interesting trend. The higher order interactions generally produce lower amplitude distortion products. In addition to the new frequencies, the amplitude of f_2 in this case is increased to $3/2$ because of the extra signal at f_2 produced by the distortion product. This is another way that these nonlinearities can appear.

RELATION TO REAL OTOACOUSTIC EMISSIONS

We have just worked through two examples of nonlinear responses where we encountered the quadratic distortion product, $(f_2 - f_1)$, and the cubic distortion product, $(2f_1 - f_2)$, common in otoacoustic emissions. One may wonder why the other products derived are not readily seen in otoacoustic emissions. We must remember that we have simplified the problem in the analysis above by neglecting the frequency response of the system. In addition, there may be higher order effects that cancel out some of the lower order distortion products, in a way opposite of the enhancement of the ω_2 term above in equation (21). The model responses that we have been playing with are by no means an accurate portrayal of the otoacoustic responses in the auditory system. Something like

$$R(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + \sin(\omega_1 t) \sin(\omega_2 t) + \sin^2(\omega_1 t) \sin(\omega_2 t) \quad (22)$$

may be a good first guess. It predicts frequencies f_1 , f_2 , $(f_2 - f_1)$, $(f_2 + f_1)$, $(2f_1 - f_2)$, and $(2f_1 + f_2)$. Why should the sine function, $\text{Sin}(\omega_1 t)$, in the cubic term be preferred (by squaring) over $\text{Sin}(\omega_2 t)$? Maybe the response should be made symmetric with respect to ω_1 and ω_2 , like this:

$$R(t) = \text{Sin}(\omega_1 t) + \text{Sin}(\omega_2 t) + \text{Sin}(\omega_1 t) \text{Sin}(\omega_2 t) + \text{Sin}^2(\omega_1 t) \text{Sin}(\omega_2 t) + \text{Sin}(\omega_1 t) \text{Sin}^2(\omega_2 t). \quad (23)$$

Perhaps one should include all the combinations:

$$\begin{aligned} R(t) = & \text{Sin}(\omega_1 t) + \text{Sin}(\omega_2 t) + && \text{(Linear Terms)} \\ & \text{Sin}^2(\omega_1 t) + \text{Sin}(\omega_1 t) \text{Sin}(\omega_2 t) + \text{Sin}^2(\omega_2 t) + && \text{(Quadratic Terms)} \\ & \text{Sin}^3(\omega_1 t) + \text{Sin}^2(\omega_1 t) \text{Sin}(\omega_2 t) + \text{Sin}(\omega_1 t) \text{Sin}^2(\omega_2 t) + \text{Sin}^3(\omega_2 t) && \text{(Cubic Terms)}. \end{aligned} \quad (24)$$

What happens in this case? (This one you can work out yourself to test your understanding.)

In a more realistic model, the terms can have different amplitudes, and these amplitudes should depend on the frequencies. Therefore, depending on the system, some distortion products will be strong whereas others will be weak or even cancel out! Try the next one in equation (25). The results are surprising.

$$R(t) = 2 \text{Sin}(\omega_1 t) - \frac{1}{2} \text{Sin}(\omega_2 t) + \text{Sin}^2(\omega_1 t) \text{Sin}(\omega_2 t). \quad (25)$$

It is similar to the cubic case that we just worked through, but this time one of the original frequencies doesn't even appear in the response!

A BIT FURTHER

What happens when there are quartic, or fourth-order terms, such as $\text{Sin}^4(\omega_1 t)$ or $\text{Sin}^2(\omega_1 t) \text{Sin}^2(\omega_2 t)$? Try working them out. For the first one treat the $\text{Sin}^4(\omega_1 t)$ as $\text{Sin}^2(\omega_1 t) \text{Sin}^2(\omega_1 t)$ and use identity (I1) to expand each quadratic term separately. Then multiply the expanded terms out to get quadratic terms with cosines. Expand these quadratic terms and you'll have the answer. The second example can be solved similarly.

One can show that any nonlinear term will result in frequencies of the form $|n_1 f_1 + n_2 f_2|$, where n_1 and n_2 are integers and the bars signify that one should take the absolute value (this avoids negative frequencies). Also, if $n_1 + n_2$ is even, then the nonlinearity is an even order (like the quadratic), and if $n_1 + n_2$ is odd, then the nonlinearity is an odd order (like the cubic).

Another interesting result is that if the ratio of the two driving frequencies, f_1/f_2 , is rational, (i.e., $f_1/f_2 = p/q$, where p and q are integers), then all of the frequencies in the response must be multiples or harmonics of $f_2 - f_1$. If the ratio of the frequencies is irrational, then $|n_1 f_1 + n_2 f_2|$ can produce all possible frequencies. It is important to remember that although all frequencies might be present in the response, the spectrum will not be smooth and continuous. Nearby frequencies may have very different amplitudes.

How do the amplitudes of the frequencies, $|n_1 f_1 + n_2 f_2|$, behave as n_1 or n_2 becomes large (for example when we have $|5f_1 + 7f_2|$)? Well, as we saw above, the quadratic term in the response function resulted in $|n_1| = 1$ and the amplitudes were only $1/2$. The cubic term resulted in $|n_1| = 1$ and $|n_2| = 2$, and the amplitudes were decreased to $1/4$. Generally, the amplitudes of the frequencies follow $\text{Exp}(-a_1 |n_1| - a_2 |n_2|)$, where a_1 and a_2 are positive numbers depending on the nonlinearity. As $|n_1|$ and $|n_2|$ get large, specifically as soon as $|n_1|$ and $|n_2|$ exceed a_1^{-1} and a_2^{-1} , the amplitude of the corresponding frequency becomes negligibly small and cannot be detected (Bergé, Pomeau, & Vidal, 1984).

We have looked at what can happen with the occurrence of two frequencies in the driving stimulus, but we have not considered interactions among three or more frequencies. As you might guess, there will be more distortion products. Specifically if r frequencies are presented, the observed frequencies will be of the form $|n_1 f_1 + n_2 f_2 + n_3 f_3 + \dots + n_r f_r|$. If the ratio of any pair of frequencies is irrational, then all frequencies can be present in the response. If all of the ratios are rational, the response will consist of multiples or harmonics of the differences between all of the pairs of driving frequencies.

We have not considered what happens with the phases of the responses. This matter is much more complicated. If we stimulate a linear system at a certain driving frequency then the system will oscillate at

that frequency. The phase of the system's oscillation depends on the resonance frequency of the system, the damping in the system, and the driving frequency. For a system with nonlinearities, the phase becomes more difficult to deal with and we will not discuss the topic in this tutorial. In the context of otoacoustic emissions, this becomes even more difficult as the travel time between the source of the emissions and the recording microphone will introduce an additional phase shift. The presence of multiple emission sites would further complicate the calculation of the phase of the response.

CONCLUSION

When a linear system is simultaneously driven at multiple frequencies, it will respond only at those frequencies. However, nonlinearities in the system, which can be described as interactions among the responses to the various frequencies present in the driving stimulus, will cause the system to respond at frequencies not necessarily present in the driving stimulus. These new frequencies are called distortion products. A quadratic nonlinearity will cause a quadratic distortion product, a cubic nonlinearity will cause a cubic distortion product, and so on. These nonlinearities can also affect the response amplitudes at the original driving frequencies. As we have seen, the distortion products are a result of the presence of nonlinearities in oscillations in general and are not restricted to biological phenomena. Therefore, one should expect that under certain conditions microphones and transducers used in the acquisition of otoacoustic emissions can generate distortion products. This occurs when they are driven outside of their linear operating range into a regime where the device behaves nonlinearly. This is an important fact to consider when performing distortion product otoacoustic emission experiments. One must be sure that the distortion products are due to the physiology of the inner ear of the subject and not due to the equipment.

Distortion products can provide much information about the nonlinearities present in a system. Although it is important to remember that a single system can produce quadratic, cubic, quartic, and higher order distortion products simultaneously, it is also important to note that there are sometimes different components of a system that are producing similar or different distortion products. For example, the presence of quadratic and cubic distortion products in an otoacoustic emission experiment does not imply that there are necessarily two distortion product sources present. One physical effect may produce both types of distortion products simultaneously. In addition, experimental manipulation of the system can change the frequency response of the system and thus affect the various orders of distortion products differently. For this reason, one must be careful not to assume that the distortion products of different orders originate from distinct sources solely on the basis that the variation of one variable in the experiment affected the distortion products differently. However, there is no reason to assume, a priori, that there is one source of nonlinearity in the inner ear. Basilar membrane motion, inner and outer hair cell motion, and efferent effects can possibly all contribute to the observed distortion products.

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REFERENCES

- Bergé, P., Pomeau Y., & Vidal C. (1984). *Order within chaos: Towards a deterministic approach to turbulence*. New York: John Wiley & Sons.
- Lins, O. G., & Picton, T. W. (1995). Auditory steady-state responses to multiple simultaneous stimuli. *Electroencephalography and Clinical Neurophysiology*, 96,420-432.