

# Measuring Questions: Relevance and its Relation to Entropy

Kevin H. Knuth

*Comp. Sci. Div., NASA Ames Research Ctr., M/S 269-3, Moffett Field CA 94035*

**Abstract.** The Boolean lattice of logical statements induces the free distributive lattice of questions. Inclusion on this lattice is based on whether one question answers another. Generalizing the zeta function of the question lattice leads to a valuation called *relevance* or *bearing*, which is a measure of the *degree* to which one question answers another. Richard Cox conjectured that this degree can be expressed as a generalized entropy. With the assistance of yet another important result from Janos Aczél, I show that this is indeed the case, and that the resulting inquiry calculus is a natural generalization of information theory. This approach provides a new perspective on the Principle of Maximum Entropy.

“A wise man’s question contains half the answer.” Solomon Ibn Gabirol (1021-1058)

## QUESTIONS AND ANSWERS

Questions and answers, the unknown and the known, empty and full are all examples of duality. In this paper, I will show that a precise understanding of the duality of questions and answers allows one to determine the unique functional form of the relevance measure on questions that is consistent with the probability measure on the set of logical statements that form their answers. Much of the material presented in this paper relies on fundamental background material that I regrettably cannot take the space to address. While I provide a brief background below, I recommend the following previous papers [1, 2, 3, 4] in which more background, along with useful references, can be found.

## LATTICES AND VALUATIONS

A partially ordered set, or poset for short, is a set of elements ordered according to a binary ordering relation, generically written  $\leq$ . One element  $b$  is said to ‘include’ another element  $a$  when  $a \leq b$ . Inclusion on the poset is encoded by the *zeta function*

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y \end{cases} \quad (\text{zeta function}) \quad (1)$$

If there is a greatest element in the poset, it is called the top  $\top$ , and dually, there may be a bottom  $\perp$ . Given two elements  $a$  and  $b$  of the poset, their upper bound is the set of all elements  $x$ , such that  $a \leq x$  and  $b \leq x$ , where  $x \neq a$  and  $x \neq b$ . If there exists a *unique*

least upper bound, this element is called the *join* of  $a$  and  $b$ , written  $a \vee b$ . Similarly, if there exists a greatest lower bound, that element is called the *meet*, written  $a \wedge b$ . A *lattice* is a poset in which unique joins and meets of all pairs of elements exist. In this case, the join and meet can be seen as binary operations that take two objects and map them to a third. For this reason, lattices are algebras. When one views the lattice as a set of elements ordered by an ordering relation, one is taking a structural viewpoint. When it is viewed as a set of elements and a set of operations, one is taking an operational viewpoint, which is an algebra. Last, elements that cannot be expressed as a join of two other elements are called *join-irreducible elements*.

Generalizing the zeta function allows one to define *degrees of inclusion*. In actuality, it is more useful to generalize the dual of the zeta function [4], which is the zeta function (1) with the conditions flipped around. The result is a real-valued function that captures the notion of degrees of inclusion

$$z(x, y) = \begin{cases} 1 & \text{if } x \geq y \\ 0 & \text{if } x \wedge y = \perp \\ z & \text{otherwise, where } 0 < z < 1. \end{cases} \quad (\text{degrees of inclusion}) \quad (2)$$

The rules by which degrees of inclusion are manipulated as one moves about the lattice are found by maintaining consistency with the lattice structure, or equivalently the underlying algebra. All lattices are associative, and as such, they all possess a sum rule. With an important result from Caticha [5], I have shown that all distributive lattices give rise [3, 4] to a sum rule,

$$z(x_1 \vee x_2 \vee \cdots \vee x_n, t) = \sum_i z(x_i, t) - \sum_{i < j} z(x_i \wedge x_j, t) + \sum_{i < j < k} z(x_i \wedge x_j \wedge x_k, t) - \cdots \quad (3)$$

a product rule

$$z(x \wedge y, t) = Cz(x, t)z(y, x \wedge t), \quad (4)$$

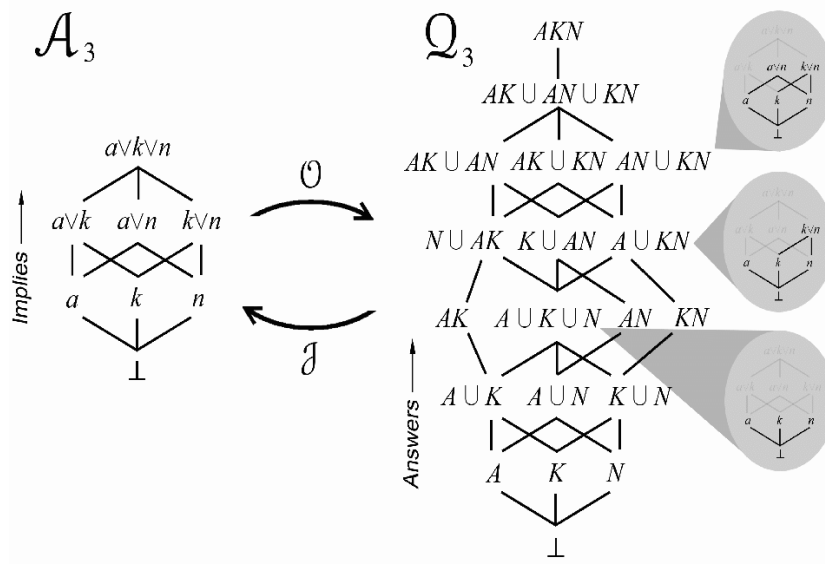
and a Bayes' Theorem

$$z(y, x \wedge t) = \frac{z(y, t)z(x, y \wedge t)}{z(x, t)}. \quad (5)$$

This immediately conjures up thoughts of probability theory, however this result is surprisingly far more general.

## PROBABILITY

It is now well-understood that probability theory is literally an extension of logic. A set of logical statements ordered by implication gives rise to a Boolean lattice, which is equivalently a Boolean algebra. Figure 1 shows the lattice  $\mathcal{A}_3$  generated from a set of three mutually exclusive and exhaustive assertions:  $a$ ,  $k$ , and  $n$ . This example is taken from two previous papers, and deals with the issue of ‘Who stole the tarts?’ in Lewis



**FIGURE 1.** The ordered set of down-sets of the lattice of assertions  $\mathcal{A}$  results in the corresponding lattice of questions  $\mathcal{Q} = \mathcal{O}(\mathcal{A})$  ordered by  $\subseteq$ .  $\mathcal{A}$  is dual to  $\mathcal{Q}$  in the sense of Birkhoff's Representation Theorem. The join-irreducible elements of  $\mathcal{Q}$  are the ideal questions  $\mathcal{J}$ , which are isomorphic to the lattice  $\mathcal{A} \sim \mathcal{J} = \mathcal{J}(\mathcal{Q})$ . Down-sets corresponding to several questions are illustrated on the right.

Carroll's *Alice in Wonderland*, specifically

- $a = \text{'Alice stole the tarts!'}$
- $k = \text{'The Knave of Hearts stole the tarts!'}$
- $n = \text{'No one stole the tarts!'}$

The lattice  $\mathcal{A}_3$  shows all possible statements that can be formed from these three atoms.

The zeta function of this lattice, which is a function of two statements, indicates whether one statement implies another. Generalizing the dual of the zeta function results in a bi-valuation  $p(x|y) \equiv z(x,y)$  that follows a *sum rule*, a *product rule* and a *Bayes' theorem*. This bi-valuation is a measure that quantifies the degree to which one statement implies another, and is essentially the degree of implication that Cox considered in his seminal work [6, 7]. Thus order theory gives rise to probability theory [3, 4].

Probability theory, however, does not instruct us on how to assign priors (which can be considered as valuations, eg.  $p(x|\top) \equiv v(x)$ ). An important theorem by Gian-Carlo Rota [8, Theorem 1, Corollary 2, p.35] makes this fact clear:

**Theorem 1** *A valuation  $v$  in a finite distributive lattice  $L$  is uniquely determined by the values it takes on the set of join-irreducibles of  $L$ , and these values can be arbitrarily assigned.*

There is no information in the Boolean algebra, and hence the inferential calculus (probability theory), to instruct us in assigning priors. We must instead rely on additional principles, such as symmetry, constraints, and consistency with other aspects of the problem to assign priors. However, once the priors are assigned, order-theoretic principles dictate the remaining probabilities through the inferential calculus.

## RELEVANCY

Cox defined a question as the set of all possible logical statements that answer it [9]. With questions being described by sets, their natural algebra is the distributive algebra, with the join  $\vee$  and meet  $\wedge$  identified with the set union  $\cup$  and set intersection  $\cap$ , respectively. The natural ordering relation among questions is the relation ‘answers’, which can be represented mathematically by  $\subseteq$ . Thus questions possess two algebraic operations  $\vee$  and  $\wedge$  analogous to the familiar disjunction and conjunction of logical statements. In fact, we even use the words ‘or’ and ‘and’ to describe them in spoken language. However, the algebra is not Boolean as Cox surmised, since the definition of a question is rather restrictive (i.e. not all sets of logical statements correspond to questions). Thus questions do not, in general, have complements [1].

In order theory, Cox’s definition of a question is equivalent to saying that a question is a down-set, where a down-set is the set of all poset elements that contain every element that includes any other element of the set [10, 4]

**Definition 1 (Down-set)** *A down-set is a subset  $J$  of an ordered set  $L$ , written  $J = \downarrow L$ , where if  $a \in J$ ,  $x \in L$ ,  $x \leq a$  then  $x \in J$ .*

where  $J$  is the question,  $L$  is the ordered set of logical statements, and  $\leq$  is ‘implies’  $\rightarrow$ . The question lattice is then formed by taking the set of down-sets of the assertion lattice and ordering them according to  $\subseteq$ . This operation, called *the ordered set of down-sets*  $\mathcal{O}$ , takes the assertion lattice to the question lattice,  $\mathcal{Q} = \mathcal{O}(\mathcal{A})$ . Figure 1 shows the lattice  $\mathcal{Q}_3$  generated from  $\mathcal{A}_3$ . The lattice  $\mathcal{Q}_3$  depicts all possible questions that can be asked in this example. Note that  $A \equiv \downarrow a$ ,  $AN \equiv \downarrow a \vee n$ ,  $AKN \equiv \downarrow a \vee k \vee n$ , and  $AN \vee AK \equiv AN \cup AK$ . The lattice  $\mathcal{A}$  is dual to the lattice  $\mathcal{Q}$  in the sense of *Birkhoff’s Representation Theorem* [10, 1], which relates a distributive lattice to its ordered set of down-sets.

The ideal questions  $\mathcal{J}$ , are the set of join-irreducible elements of  $\mathcal{Q}$ . They are not practical questions, but are useful mathematical constructs since they are isomorphic to the assertion lattice. The real questions  $\mathcal{R}$  are the set of all questions that can be answered by each of the atomic statements  $a$ ,  $k$ , or  $n$ . The question  $I = A \vee K \vee N$  is a special real question that I call the *central issue* [4]. It is the unique real question that answers all the others. In this example, it asks ‘Precisely who stole the tarts?’

Just as in the lattice of assertions, we can define the degree to which one question answers another by  $d(X|Y) = z(X, Y)$ . Since the lattice of questions is distributive, there exists a sum rule, a product rule, and a Bayes’ theorem. This degree is called relevance, and due to the duality between  $\mathcal{A}$  and  $\mathcal{Q}$  it is entirely reasonable to expect that relevance on  $\mathcal{Q}$  is related to probability on  $\mathcal{A}$ . We explore this in the next section.

## CONSISTENCY BETWEEN PROBABILITY AND RELEVANCY

The sum, product and Bayes’ rules ensure consistency *within* the assertion and question lattices, however our assignments of probabilities and relevances must also be mutually consistent with one another. Rota’s theorem assures that we need only to determine the relevances of the join-irreducible questions; the rest follow from the inquiry calculus.

In this section, I show that the form of the relevance is uniquely determined by requiring consistency between the probability measure defined on the assertion lattice  $\mathcal{A}$  and the relevance measure defined on its isomorphic counterpart, the lattice of ideal questions  $\mathcal{J}$ . This demonstration requires but a single assumption: the degree to which the top question  $\top$  answers an ideal question  $X$  depends only on the probability of the assertion  $x$  from which the question  $X$  originated. That is, given the ideal question  $X = \downarrow x$

$$d(X|\top) = H(p(x|\top)), \quad (6)$$

where  $H$  is a function to be determined.

There are four important consistency requirements imposed by the lattice structure and the induced calculus. First, the sum rule (3) for questions demands that given three questions  $X, Y, Q \in \mathcal{Q}$  the relevance is *additive* only when  $X \wedge Y = \perp$

$$d(X \vee Y|Q) = d(X|Q) + d(Y|Q), \quad \text{iff } X \wedge Y = \perp. \quad (\text{additivity}) \quad (7)$$

and is *subadditive*

$$d(X \vee Y|Q) \leq d(X|Q) + d(Y|Q). \quad (\text{subadditivity}) \quad (8)$$

in general; a result of the terms in the sum rule (3), which avoid double-counting the overlap between the two questions [3, 4]. Commutativity of the join requires that

$$d(X_1 \vee X_2 \vee \cdots \vee X_n|Q) = d(X_{\pi(1)} \vee X_{\pi(2)} \vee \cdots \vee X_{\pi(n)}|Q) \quad (\text{symmetry}) \quad (9)$$

for all permutations  $(\pi(1), \pi(2), \dots, \pi(n))$  of  $(1, 2, \dots, n)$ . Thus the relevance must be *symmetric* with respect to the order of the joins.

Last, since any assertion  $f$ , known to be false can be identified with the bottom  $\perp$  in  $\mathcal{A}$ , its corresponding ideal question  $F = \downarrow f \in \mathcal{J}$  can be identified with  $\perp$  in  $\mathcal{Q}$ . Since for all questions  $X \in \mathcal{Q}$  it is true that  $X \vee \perp = X$ , we have the *expansibility* condition

$$d(X_1 \vee X_2 \vee \cdots \vee X_n \vee F|Q) = d(X_1 \vee X_2 \vee \cdots \vee X_n|Q). \quad (\text{expansibility}) \quad (10)$$

I now define a *partition question* as a real question where its set of answers are neatly partitioned. More specifically

**Definition 2 (Partition Question)** A partition question is a real question  $P \in \mathcal{R}$  formed from the join of a set of ideal questions  $P = \bigvee_{i=1}^n X_i$  where  $\forall X_j, X_k \in \mathcal{J}(\mathcal{Q}), X_j \wedge X_k = \perp$  when  $j \neq k$ .

For a partition question  $P$ , the degree to which the top question  $\top$  answers  $P$  can be easily written using (7)

$$d(P|\top) = d\left(\bigvee_{i=1}^n X_i|\top\right) = \sum_{i=1}^n H(p(x_i|\top)). \quad (11)$$

An important result from Aczél et al. [11] states that if a function of this form satisfies additivity (7), subadditivity (8), symmetry (9), and expansibility (10), then the unique form of the function is a linear combination of the Shannon and Hartley entropies

$$d(P|\top) = a H_m(p_1, p_2, \dots, p_n) + b {}_oH_m(p_1, p_2, \dots, p_n), \quad (12)$$

where  $p_i \equiv p(x_i|\top)$ ,  $a, b$  are arbitrary non-negative constants. The Shannon entropy [12] is defined as

$$H_m(p_1, p_2, \dots, p_n) = - \sum_{i=1}^n p_i \log_2 p_i, \quad (13)$$

and the Hartley entropy [13] is defined as

$${}_oH_m(p_1, p_2, \dots, p_n) = \log_2 N(P), \quad (14)$$

where  $N(P)$  is the number of non-zero arguments  $p_i$ . An additional condition suggested by Aczél states that the Shannon entropy is the unique solution if the result is to be small for small probabilities [11]; that is, the relevance varies continuously as a function of the probability. This result is important since it rules out the use of other types of entropy, such as the Renyi and Tsallis entropies, for the purposes of inference and inquiry.

Given these results, the relevance of an ideal question (6) can be written as

$$d(X|\top) = -ap(x|\top) \log_2 p(x|\top), \quad (15)$$

which is proportional to the probability-weighted surprise. The sum rule allows us to calculate more complex relevances, such as that of the central issue

$$d(A \vee K \vee N|\top) \propto -p_a \log_2 p_a - p_k \log_2 p_k - p_n \log_2 p_n, \quad (16)$$

where  $p_a \equiv p(a|\top), \dots$ , and we have set the arbitrary constant  $a = 1$ .

With the relevances of the join-irreducible questions defined, the inquiry calculus allows us to compute the relevance between any two questions. The degree to which an arbitrary question  $Q$  answers a question  $X$  can be found from  $d(X|\top)$  by recognizing that  $d(X|Q) = d(X|Q \wedge \top)$  and using Bayes' Theorem. Furthermore, the relevance of  $Q$  to the join of two questions such as  $AN \cup KN \equiv AN \vee KN$  is

$$d(AN \vee KN|Q) = d(AN|Q) + d(KN|Q) - d(AN \wedge KN|Q), \quad (17)$$

which is clearly related to the mutual information, although the conditionality of this measure absent in the information-theoretic notation. Thus the relevance of the join of two questions is akin to mutual information, which describes what the two questions ask in common. Similarly, the relevance of the meet of two questions  $d(AN \wedge KN|Q)$  is akin to the joint entropy. In the context of information theory, Cox's choice in naming the common question and joint question is very satisfying.

The inquiry calculus holds new possibilities. Not only does it allow for conditionality, which is obscured and implicit in information theory, but the relevance of questions comprised of the joins of multiple questions can be computed using the sum rule, which proves to be the *generalized entropy* conjectured by Cox [7, 9]. Furthermore, special cases of these relevances have appeared before in the literature [4]. The result presented here is a well-founded generalization of information theory, where the relationships among a set of any number of questions can be quantified.

Last, it should be noted that by setting  $a = b$  in (12), and using (13), (14), we get

$$H_m(p_1, p_2, \dots, p_n) = - \sum_{i=1}^n p_i \log_2 \frac{p_i}{\frac{1}{n}}, \quad (18)$$

which is the relative entropy based on a uniform measure.

## MAXIMUM ENTROPY

This result provides new insights into the assumptions underlying the *Principle of Maximum Entropy* [14, 15]. Consider both the assertion lattice  $\mathcal{A}$  and its dual the question lattice  $\mathcal{Q}$ . What does it mean to assign probabilities to  $\mathcal{A}$  by maximizing the entropy? When we ‘maximize the entropy’, we are actually maximizing the relevance of the top question  $\top$  to the central issue  $I$ , i.e. we maximize  $d(I|\top)$ . This says that we are setting up the probability assignments so that the question that asks everything is maximally relevant to the central issue. To understand what this means, it is useful to see what happens in a special case. In the situation where we have no constraints, this results in assigning uniform prior probabilities to the join-irreducible elements of  $\mathcal{A}$ . What if in the case of three statements, we assign the probabilities non-uniformly:  $p(x_1|\top) = 0.1$ ,  $p(x_2|\top) = 0.4$ ,  $p(x_3|\top) = 0.5$ . In this case, the central issue no longer has the maximal relevance. Instead, the question defined by the set  $X_1X_2 \vee X_3 = \{x_1 \vee x_2, x_1, x_2, x_3, \perp\}$  has the maximal relevance. This suggests that a re-parametrization of the problem is more *relevant*  $\{x_1 \vee x_2, x_3\}$ . Thus when we assign priors based on maximizing the entropy, we are relying on the fact that we believe that we have a relevant parametrization of the problem. In other words: we have identified the *relevant variables*.

## DISCUSSION

Within the last few years there has been a surge of interest in Bayesian methods. Much of this is due to the fact that Bayesian methods work, and work well. However, with this surge of activity the ideas of Jaynes and Cox are slowly being lost as converted statisticians focus more and more on mathematical rigor and less on the basic concepts. Ironically, it is this focus on mathematical rigor and loss of the basic concepts that buried Bayesian methods in the 19th century. Cox’s realization that Bayesian probability theory is the only theory consistent with Boolean logic is key, since it rules out all other possible theories of inference. Jaynes’ realization that the entropy of statistical mechanics is related to Shannon’s entropy, and that one can use it to assign priors is crucial since it ties together the physics of thermodynamics to inference. However, the successful application of inference in several key areas of physics seems to be of little interest to statisticians, which is puzzling given both the great success of the theories and the great mysteries that they simultaneously resolve and reveal.

The basic concepts are key, because it is by fully understanding these concepts that we can generalize these ideas to form new theories. I have found that Cox’s idea of introducing a real number representing a degree of belief can be generalized to introducing a real-valued function generalizing the zeta function of a lattice. This allows one to take any ordered set that forms a lattice and introduce a measure describing the degree of inclusion associated with that ordering relation.

Cox’s definition of a question is the definition of a down-set in order theory. With this definition in hand, I showed that the ordered set of down-sets of assertions gives rise to the set of all possible questions, which forms a distributive lattice. Realizing that Caticha’s results on quantum mechanical experimental setups demonstrate that sum

and product rules are associated with distributive lattices, I showed that the calculus of inquiry has sum and product rules analogous to the inferential calculus. This paper extends these results by requiring consistency between the measures assigned to the lattice of assertions and the lattice of questions. With yet another important result by Aczél, I have shown that the relevance measure on the lattice of questions is based on the Shannon entropy. This is significant since it rules out the use of other entropies in inference (eg. Renyi entropy and Tsallis entropy), as well as inquiry. The result is that the inquiry calculus and the relevance measure is a natural generalization of information theory. Furthermore, these results provide a new perspective on the role of Maximum Entropy in prior probability assignment.

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